Exploring Causal Relationships in the Phase Functions of Coupled Van Der Pol Oscillators

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Abstract. We investigate whether causal relationships in coupled Van der Pol oscillators can be determined. Using the instantaneous phase of computer generated time series, which are unidirectionally and time dependently coupled, the technique of lagged mutual information is tested to determine its usefulness in detecting information transmission. If successful, it could assist in improving our understanding of the role of information transmission in organizing CNS activity.

Introduction:

The study of Van der Pol oscillators has been of interest since Balthasar van der Pol's work on oscillations in a triode circuit led to the development of the Van Der Pol equation[1],[2]. The equation of motion for the Van der Pol oscillator can be written as,

$$\ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + \omega_0 x = 0 \tag{1}$$

The linear term represents a constant force which determines the natural frequency. The self-sustained oscillations are modified by the nonlinear damping term. There is positive damping or energy is dissipated from the system when $|x| > 1$, and there is negative damping or energy is added to the system when $|x| < 1$.

Rewriting (1) as first order equations, the coupled system being studied is,

System 1 $\frac{dx_1}{dt} = y_1$ **System 2** $\frac{dx_2}{dt} = y_2$

(2)

$$\frac{dy_1}{dt} = -\mu_1\left(1 - \hat{x}_1^2\right)y_1 - \omega_{01}x_1$$
$$\frac{dy_2}{dt} = -\mu_2\left(\hat{x}_2^2 - 1\right)y_2 - \omega_{02}x_2$$

where $\hat{x}$ is the coupling term, and ε is the coupling parameter;

$$\hat{x}_1 = \varepsilon_{2,1}x_2 + (1-\varepsilon_{2,1})x_1$$
$$\hat{x}_2 = \varepsilon_{1,2}x_1 + (1-\varepsilon_{1,2})x_2 \qquad (3)$$

Physically, this means that the two oscillators are coupled only through the damping term. For $\varepsilon_{1,2} = \varepsilon_{2,1} = 0$, we have two uncoupled oscillators whose limit cycles are determined by the damping constants μ_1 and μ_2, and the natural frequency terms ω_{01} and ω_{02}.

Although there are different methods for calculating the phase of a signal, we chose an approach based on the Hilbert Transform. This technique was originally applied by Gabor[3]. For an arbitrary signal s(t), an analytic signal of the form,

$$\Psi(t) = s(t) + s_{SH}(t) = A(t)e^{i\phi(t)} \qquad (4)$$

is defined, where $s_{SH}(t)$ is the Hilbert Transform of s(t), A(t) is the instantaneous amplitude, and $\varphi(t)$ is the instantaneous phase[4].

A fourth order Runge-Kutta algorithm was used for the numerical integration of the system equations (2). The parameters chosen were ε_{MAX} (t) = 0.3, $\mu_1 = 0.5$, $\mu_2 = 2$, $\omega_{01} = 7$, $\omega_{02} = 11$, the time step = 0.1, and the equations were integrated for 5000 steps. Since we are interested in detecting causal relationships, the coupling terms ε (t), was unidirectional and time dependent. For the time steps, 1 – 1000, 2001 – 3000, and 4001 – 5000, there was no coupling, i.e., $\varepsilon_{1,2} = \varepsilon_{2,1} = 0$. For the interval 1001-2000, $\varepsilon_{1,2} = 0.3$, $\varepsilon_{2,1} = 0$; that is, the position of system 1 was coupled to the damping term of system 2. Finally, for interval 3001 – 4000, we set $\varepsilon_{1,2} = 0$, and $\varepsilon_{2,1} = 0.3$.

Causal Relationships:

While synchronization measures can be used to prove a correlative relationship between two signals, they cannot establish causal relationships or provide any information concerning the direction or flow of information. In most cases, a quantitative assessment of causal relationships between two systems is built on the following idea. If the measurement of variable X improves the prediction of variable Y, then Y is, in at least this limited operational sense, causally dependent on X. It should be stressed that this relationship is not necessarily unidirectional. It can also be the case that, with the same data, measuring variable Y also improves the prediction of variable X. (This conceptualization of causality appears in Wiener[5].

An early implementation of this concept was published by Granger[6] in the econometrics literature and popularized by Sims[7]. Granger causality is constructed using linear regression models of stochastic processes. In the sense operationalized by Granger, if past values of X are useful in predicting the current value of Y in a linear regression, then X has a causal relationship with time series Y. There are several variants of this concept. A complementary procedure for the investigation of causal relationships can be constructed by examining delayed mutual information functions.

Lagged Mutual Information

The mutual information of time series X and Y, I(X,Y) is defined as the average number of bits of one variable that can be predicted by measuring the other. Mutual information is symmetrical, that is, I(X,Y) = I(Y,X). Therefore, while mutual information can establish the presence of a nonrandom relationship between time series, it cannot identify causal relations. However, a time lagged mutual information in which one of the two variables is time shifted can be used to determine if, for example, measuring variable X in the past allows prediction of future values of variable Y. We can shift time series X by some time lag τ and calculate $I(X(\tau),Y)$ as a function of τ. Similarly, we can calculate $I(X,Y(\tau))$.

If measuring $X(\tau)$ allows better prediction of Y, than the other way around, then it can be argued that information is transferred from X to Y. The magnitude of the mutual information and the time lag which produces the greatest value of I(X,Y) can be used to quantify both the magnitude of the information transfer and the time delay associated with that transfer. A number of investigators have proposed using lagged mutual information to investigate information transfer in distributed systems[8],[9]. In previous work[10], found that information transfer as determined by lagged mutual information calculated between a scalp electrode located near an epileptogenic focus and other electrodes of the montage increased prior to seizure onset.

Conversely, significant limitations of causality measures based on lagged mutual information have been identified in Schreiber[11]. He argues that "time delayed mutual information fails to distinguish between information that is exchanged from shared information due to common history and inputs." Since that time there have been some significant improvements in the calculation of mutual information, specifically we refer to the development of the CAR algorithm[12]. Because some of the deficiencies of prior methods have been overcome, we believe that a second look at lagged mutual information as a casualty measure is warranted.

After solving the system equations (2) using the parameter values given above, the instantaneous phase for each time series was generated using the Hilbert Transform. Five hundred point epochs were used in calculating the mutual information, and the center of each time interval was used for the temporal location of each epoch. One epoch remains constant while the other epoch is moved forward in time. Past values of the mutual information of one system could then be compared with future values of the mutual information of the other system. Our results are summarized in Figure 1.

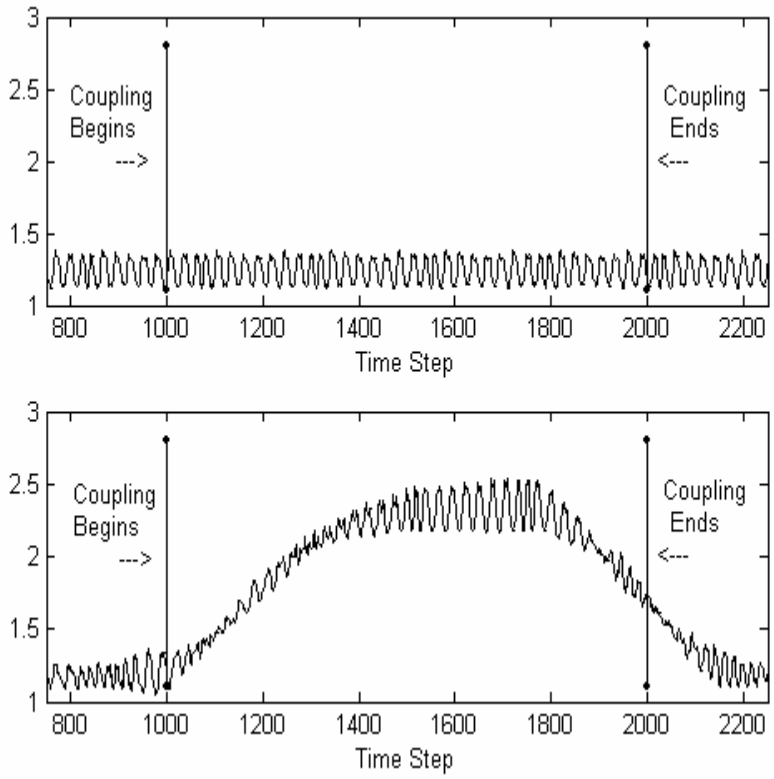

Fig. 1. (*Upper plot*) is the mutual information between a 500 point epoch of System 1 centered at time step 1000 and moving forward in time to a position centered at time step 2000, and a stationary 500 point epoch of System 2 centered at time step 1000. Since there is no coupling from System 2 to System 1 we see no change in mutual information. (*Lower plot*) is the mutual information between a 500 point epoch of System 2 centered at time step 1000 and moving forward in time to a position centered at time step 2000, and a stationary 500 point epoch of System 1 centered at time step 1000. There is a significant change in mutual information over the coupling interval. Comparable results were obtained for the time step interval 3001-4000, where System 2 was coupled to System 1.

Conclusions

These preliminary results suggest that there may still be some potential in the use of lagged mutual information as a casualty measure. Our results showed that an increase in mutual information corresponded in time with unidirectional coupling, and that the increase decayed when the coupling ended. Future questions to be addressed should include determining the noise robustness of this algorithm. Biological signals, particularly EEG's are generally noisy, so methods whose performance deteriorates significantly with noise are of limited use.

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